A Read-Through of Stokey-Zeckhauser Example (pp. 180ff)

Two activities

The Matrix Version of Linear Programming

In general, we will be dealing with systems in which we have lots of variables that can influence an outcome we are interested in. To qualify for a linear programming approach, these need to be able to be related to the outcome "linearly." This means the influence of a change in the variable is proportional to that change (and not to it's square root or it's cube):

Effect on outcome = proportionality constant x change in variable

The following are linear expressions

$$y = mx + b$$

$$income = 23 \times years\_of\_education\ + 30$$

$$degrees\_centigrade\ = (degrees\_fahrenheit - 32) \times \frac{5}{9}$$

easier to see if we rearrange.

$$degrees\_centigrade\ = \frac{5}{9} \times degrees\_fahrenheit - \frac{5 \times 32}{9}$$

$$y = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$$

This last one is the most general. It has two parts. The first is a list of "coefficients" (the m's) which tell us how much each variable contributes to our outcome. The second is the list of variables (the x's). The equation says "you can compute the outcome (y) by taking the list of variable values and multiplying each one by the appropriate coefficient and then adding the products together":

Let's represent the c's like this $(c_1 \quad c_2 \quad c_3 \quad c_4)$ and the x's like this $(x_1 \quad x_2 \quad x_3 \quad x_4)$. We call a set of numbers like this (it's basically an extension of an "ordered pair" (x,y) a "vector"[1] and often you'll see (or hear) the term n-vector meaning a vector with n elements. We can then write the above equation like this:

$$y = (c_1 \quad c_2 \quad c_3 \quad c_4)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

This multiplication happens element by element scanning across the first vector and down the second adding the products as we go (c_1 times x_1 plus c_2 times x_2 etc.). We'll later see why this rule makes sense (that is, why we can't just multiply two vertical vectors together) – for now, just take it as the way it's done.

By convention we represent vectors by bold printed letters like this:

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1 You may associate "vector" with an arrow pointing in a given direction or, if you've studied epidemiology, with the thing that carries a disease. Both of these are associated with motion from one place to another and we are all familiar with the most common way to specify such motion: how far do you need to move and in what direction (as given, say, by a longitude and latitude).

$$\boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

To get a vector written vertically into horizontal form (or vice versa) we "transpose" it and this is represented by a superscript T:

$$\boldsymbol{c^T} = (\boldsymbol{c_1} \quad \boldsymbol{c_2} \quad \boldsymbol{c_3} \quad \boldsymbol{c_4})$$

Thus, our equation looks like this:

$$\boldsymbol{c^T x} = \boldsymbol{c}_1 x_1 + \boldsymbol{c}_2 x_2 + \boldsymbol{c}_3 x_3 + \boldsymbol{c}_4 x_4$$

Now let's look at our constraints. Typically, we have one or more linear equations relating one or more of our variables to a constant. Let's assume we have three constraints and four variables. These would look like this

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + \leq b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 + \leq b_3$$

The "a" coefficients are double subscripted. The first subscript corresponds to which constraint equation we are in and the second one tells us which variable the coefficient goes with. The b's are the numbers that are a part of each constraint. The a-coefficients can be pulled out of each of these equations just like we did above as an "A vector times an X vector":

$$(a_{11} \quad a_{12} \quad a_{13} \quad a_{14}) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \leq b_1$$

We can stack the four A vectors together in what we call a matrix:

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

And we can write the b's as

$$\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

And so we can write the constraints as

$$\boldsymbol{Ax} \leq \boldsymbol{b}$$

STANDARD FORM

So the standard maximization problem can be stated like this. Find an n-vector $x = (x_1, \dots, x_n)^T$ to maximize

$$\boldsymbol{c}^T\boldsymbol{x} = c_1x_1 + \cdots + c_nx_n$$

subject to the constraints

$$\boldsymbol{Ax} \le \boldsymbol{b}$$

and

$$\boldsymbol{x} \ge \boldsymbol{0}$$